Ratios, Rates, and Proportions

LEARNING OBJECTIVES

1. Write ratios/rates as factions in simplest form.
2. Find unit rates.
3. Determine if a proportion is true.
4. Solve proportions.
5. Solve application problems with proportions.
6. Solve problems involving similar figures with proportions.

Ratios and Rates

RATIOS are used, typically, to compare two like quantities. For example, if there are 13 males and 17 females, then the ratio of males to females is 13 to 17.

How to write a ratio.
Ratios can be written in three different ways. In above example of 13 males to 17 females,

1. In words: 13 to 17.
2. As a fraction: $\frac{13}{17}$.
3. Using a colon: 13 : 17

In all three cases, we read "13 to 17".

Example 1

During one particular month, the number of sunny days in Bellingham was 4 days while the number of rainy days was 12. Write the ratio of number of sunny days to number of rainy days.

4 to 12 or 4:12 or $\frac{4}{12} = \frac{1}{3}$

As you can see in the above example, since you can write any ratio as a fraction, ratio can be written in simplest form by reducing the fraction.

Example 2

A picture on a wall has length 32 inches and width 12 inches. Write the ratio of length to width in simplest from.

$$\frac{32}{12} = \frac{8}{3}$$

So, the ratio is:

8 to 3 or 8:3 or $\frac{8}{3}$

Now suppose that in example 2, the measurements were given in different units. Consider the following example.

Example 3

A picture on a wall has length 2 feet and width 15 inches. Write the ratio of length to width in simplest from.

Note that in order for us to compare these two measurements and to form a correct ratio, we must have the same units for both quantities. Since 2 feet = 24 inches, length to width ratio is given by

$$\frac{24}{15} = \frac{8}{5}$$

So, the ratio is:

8 to 5 or 8:5 or $\frac{8}{5}$

Try this! Write each ratio in simplest form

a. 42 lbs to 63 lbs

b. 2 weeks to 10 days

Answer:

a. 2 to 3 (or 2:3 or $\frac{2}{3}$)

b. 7 to 5 (or 7:5 or $\frac{7}{5}$)

RATES are used to compare two unlike quantities with different units. For example, if a car is driven for 385 miles on 14 gallons of gas in the tank, then we can describe the comparison of these two quantities as $\frac{385\ \text{miles}}{14\ \text{gallons}}$. Note that the numerator and the denominator have different units. Just as ratios are simplified to lowest terms, so are rates. The rate of $\frac{385\ \text{miles}}{14\ \text{gallons}}$ can be written in simplest form as $\frac{55\ \text{miles}}{2\ \text{gallons}}$ by reducing the fraction. By performing the division, $55 \div 2$, we get what is called a unit rate. In this example since $55 \div 2 = 27.5$ We can describe the same rate as a unit rate of 27.5 miles per gallon. Unit rates are useful in comparing various rates as you will see in an example given below.

Example 4

Write each rate 26 cups for 8 people as a fraction in simplest form, then give the unit rate.

Solution:

$$\frac{26}{8} = \frac{13}{4}$$

So, the rate in simplest form is $\frac{13\ \text{cups}}{4\ \text{people}}$.

Since $26 \div 8 = 13 \div 4 = 3.25$, the unit rate is 3.25 cups per person.

Try this! Write each rate in simplest form, then give the unit rate.

a. 30 pencils for 12 people

b. $27 for 6 lbs of almonds

c. 46 hours in 8 weeks

d. 28,770 new jobs created in 60 months

Answer:

a. $\frac{5\ \text{pencils}}{2\ \text{people}}$, 2.5 pencils per person

b. $\frac{\$9}{2\ \text{lbs}}$, $4.5 per pound

c. $\frac{23\ \text{hours}}{4\ \text{weeks}}$, 5.75 hours per week

d. $\frac{959\ \text{jobs}}{2\ \text{months}}$, 479.5 jobs per month

Proportitons

Two ratios (or two rates) in fractions set equal to each other is called a proportion.

A proportion is true if those two fractions set equal are equivalent. Otherwise, we say that the proportion is false.

Example 5

These are proportions:

a. $\frac{4}{5} = \frac{20}{30}$

b. $\frac{2}{3} = \frac{8}{12}$

c. $\frac{7}{8} = \frac{28}{30}$

Note that of the three proportions above, only b is true since $\frac{2}{3}$ and $\frac{8}{12}$ are equivalent fractions.

How do you know if a proportion is true?

A proportion of the form,

$$\boldsymbol{\frac{a}{b} = \frac{c}{d}}$$

is true if $\boldsymbol{ad = cd}$. In other words, given a proportion, you can check if it is true by cross-multiplication.

To check if $\frac{a}{b} = \frac{c}{d}$ is a true proportion, cross multiply!

$$\frac{a}{b} = \frac{c}{d}$$

$b \cdot c$ $a \cdot d$

Is $ad = bc$? If yes, then it is a true proportion!

Example 6

Are these true proportions?

a. $\frac{5}{7} = \frac{35}{49}$ b. $\frac{13}{15} = \frac{52}{60}$ c. $\frac{6}{7} = \frac{35}{42}$

a. True since $5 \cdot 49 = 7 \cdot 35 = 245$
b. True since $13 \cdot 60 = 15 \cdot 52 = 780$
c. False since $6 \cdot 42 = 252 \neq 245 = 7 \cdot 35$

Try this! Determine if each of the following is a true proportion.

a. $\frac{11}{13} = \frac{63}{78}$ b. $\frac{3}{4} = \frac{81}{108}$ c. $\frac{5}{8} = \frac{85}{136}$

Answer:
a. False b. True c. True

Solving proportions

Solving a proportion means to find an unknown quantity within a proportion. We can do so using cross-multiplication. As an example, let's solve the proportion,

$$\frac{7}{x} = \frac{91}{104}$$

In order for this proportion to be true, we must have $91x = 7 \cdot 104$. So,

$$91x = 728$$

Solving this equation by dividing both sides by 91, we find that $x = 8$.

Example 7

Solve each of the following proportions.

a. $\frac{2}{x} = \frac{5}{12}$

Cross-multiply!

$2 \cdot 12 = 5x$
$24 = 5x$
$x = \frac{24}{5}$ or 4.8

b. $\frac{x+1}{5} = \frac{1}{4}$

Cross-multiply!

$4(x + 1) = 5 \cdot 1$
$4x + 4 = 5$
$4x = 1$
$x = \frac{1}{4}$ or 0.25

c. $\frac{2}{3} = \frac{4x}{x-3}$

Cross-multiply!

$2(x - 3) = 12x$
$2x - 6 = 12x$
$-6 = 10x$
$x = -\frac{6}{10} = -\frac{3}{5}$ or -0.6

Applications

Proportions can be used to solve many application problems. Consider the following example.

Example 8

The distance between City A and City B is 270 miles. On a certain map, this distance is scaled down to 4.5 inches. If the distance between City B and City C on the same map is 12 inches, what is the actual distance between City B and City C?

Solution:
Let x be the actual distance between City B and City C in miles. Since the ratio of actual distance to the distance on the map should be the same between any two cities, we can set up a proportion as follows:

$$\frac{\mathbf{270\ miles}}{\mathbf{4.5\ inches}} = \frac{x\ \mathbf{miles}}{\mathbf{12\ inches}}$$

By cross-multiplying, we get:

$$4.5x = 270 \cdot 12$$
$$4.5x = 3240$$

So… $x = 720$. The actual distance between City B and City C is 720 miles.

Try this!

Suppose you want to enlarge a 4-inch by 6-inch photograph to a poster. If you want the shorter edge of the enlarged poster to be 26 inches, how long is the longer edge of the poster?

Answer: 39 inches

Similar figures

Similar figures are figures whose corresponding sides are proportional. What does this mean?

For example,

Figure 1

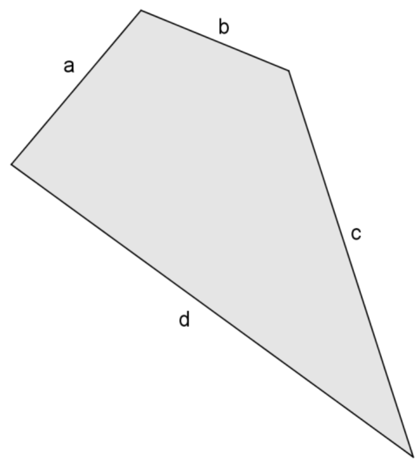


Figure 2

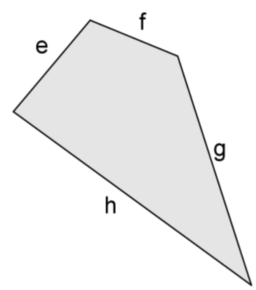
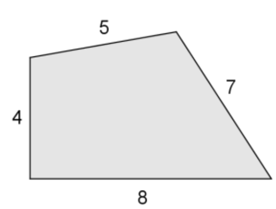


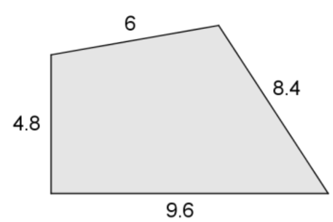
Figure 1 and Figure 2 are similar figures if the corresponding sides are proportional. That is, these are similar if the following relationship is true.

$$\boldsymbol{\frac{a}{e} = \frac{b}{f} = \frac{c}{g} = \frac{d}{h}}$$

If any one of the fraction fails to equal the others, then they are not similar.

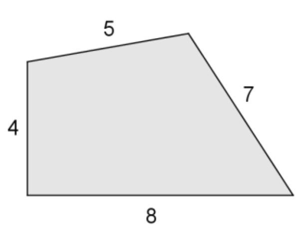
In short…
Similar figures

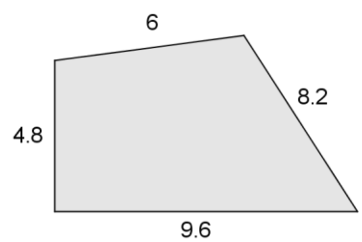




$$\boldsymbol{\frac{4.8}{4} = \frac{6}{5} = \frac{8.4}{7} = \frac{9.6}{8} = 1.2}$$

Not similar figures



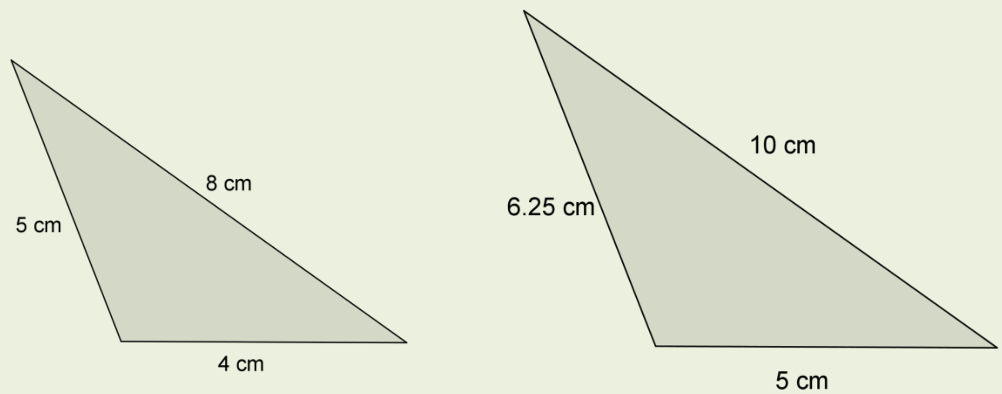


$$\boldsymbol{\frac{4.8}{4} = \frac{6}{5} = \frac{9.6}{8} = 1.2 \neq \frac{8.2}{7}}$$

Example 9

Are these similar figures?

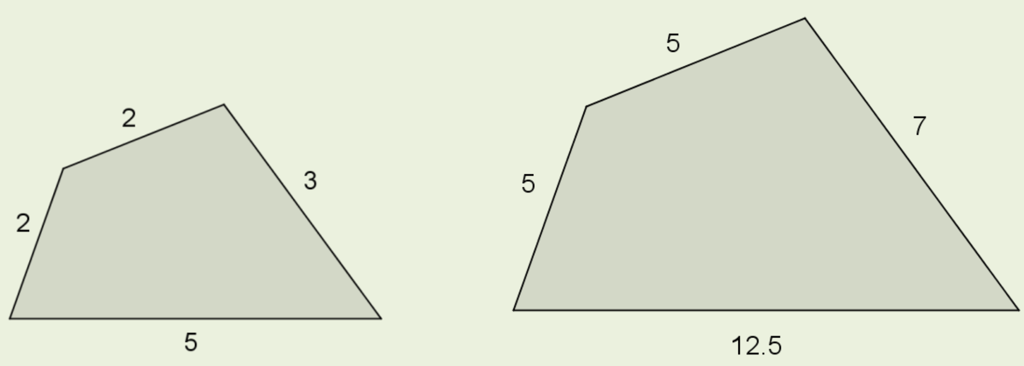
1.



Note that the ratio of the bases of these triangles is $\frac{4}{5}$. If the other two pairs have the same ratio, then these are similar figure. We see that $\frac{4}{5} = \frac{8}{10}$ (how do you know this? Check!) And $\frac{4}{5} = \frac{5}{6.25}$ (check this too!). So… $\frac{4}{5} = \frac{8}{10} = \frac{5}{6.25}$.

Since corresponding sides are proportional **these are similar!**

2.

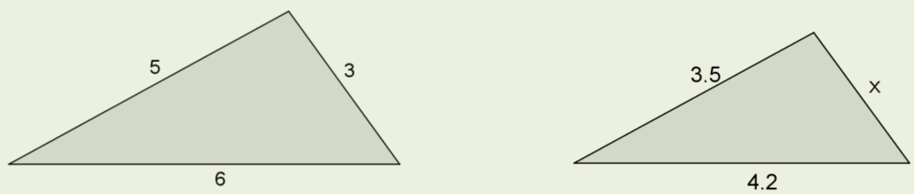


Note that one of the corresponding pair (left side) has the ratio $\frac{2}{5}$.
And we also have $\frac{2}{5} = \frac{5}{12.5}$ (check!) However, $\frac{2}{5} \neq \frac{3}{7}$ (check!).

So, **these two figures are not similar.**

Example 10

Given that following figures are similar, find $\boldsymbol{x}$.



Since corresponding sides are proportional we can write:

$$\frac{\boldsymbol{x}}{\mathbf{3}} = \frac{\mathbf{4.2}}{\mathbf{6}} \quad \textbf{or} \quad \frac{\boldsymbol{x}}{\mathbf{3}} = \frac{\mathbf{3.5}}{\mathbf{5}}$$

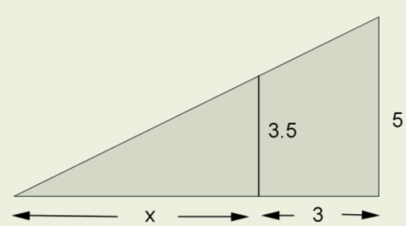
Let's solve the first one to find $\boldsymbol{x}$. As we learned earlier, we can cross multiply to get

$$\mathbf{6}\boldsymbol{x} = \mathbf{4.2} \cdot \mathbf{3}$$
$$\mathbf{6}\boldsymbol{x} = \mathbf{12.6}$$
$$\boldsymbol{x} = \mathbf{2.1}$$

How about this problem?

Example 11

Find $\boldsymbol{x}$.



Do you see similar triangles in this diagram? There are two triangles; small one within the larger one. Do you see them? The smaller one's base is $\boldsymbol{x}$ while the base of the larger one is $\boldsymbol{x} + \mathbf{3}$. The height of the smaller triangle is 3.5 whereas the height of the larger one is 5. We can use these two similar triangles to set up a proportion as follows.

$$\frac{\boldsymbol{x}}{\boldsymbol{x} + \mathbf{3}} = \frac{\mathbf{3.5}}{\mathbf{5}}$$

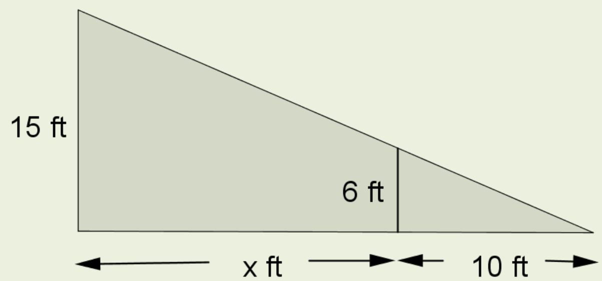
By cross-multiplying you will get $\mathbf{5}\boldsymbol{x} = \mathbf{3.5}(\boldsymbol{x} + \mathbf{3})$. By solving this equation you will see that $\boldsymbol{x} = \mathbf{7}$.

Now let's do some applications.

Example 12

A man 6 feet tall is standing near a street lamp that is 15 feet tall. If his shadow casted on the ground is 10 feet, how far away is this man standing from the lamp?

You can use similar figures to solve this problem. First, let's draw a diagram.



In this diagram, the street lamp is being represented by the height of the larger triangle, the height of the smaller triangle represents the 6-feet tall man. $\boldsymbol{x}$ represents the distance between the man and the street lamp. Using this diagram, we can set up a proportion like we did in **Example 11** as follows.

$$\frac{\mathbf{6}}{\mathbf{15}} = \frac{\mathbf{10}}{\boldsymbol{x} + \mathbf{10}}$$

By cross-multiplying, we get $\mathbf{6(\boldsymbol{x} + 10) = 10 \cdot 15}$. By solving this equation, we see that $\boldsymbol{x} = \mathbf{15}$. So the man is standing 15 feet from the street lamp.